Ann. Combin. 2025

Increasing Binary Trees and the (α, β)-Eulerian Polynomials

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Abstract

In light of the grammar given by Ji for the (α, β)-Eulerian polynomials introduced by Carlitz and Scoville, we provide a labeling scheme for increasing binary trees. In this setting, we obtain a combinatorial interpretation of the γ-coefficients of the α-Eulerian polynomials in terms of forests of planted 0-1-2-plane trees, which specializes to a combinatorial interpretation of the γ-coefficients of the derangement polynomials in the same spirit. By means of a decomposition of an increasing binary tree into a forest, we find combinatorial interpretations of the sums involving two identities of Ji, one of which can be viewed as (α, β)-extensions of the formulas of Petersen and Stembridge.

Keywords: Context-free grammars, grammatical labelings, increasing binary trees, (α, β)-Eulerian polynomials, γ-positivity.

AMS Classification: 05A15, 05A19

1 Introduction

The objective of this paper is to explore a labeling scheme for increasing binary trees as an alternative combinatorial interpretation of the (α,β)-Eulerian polynomials introduced by Carlitz and Scoville [2]. A grammatical treatment of these polynomials has been given by Ji [6] via a labeling scheme for permutations. Employing the grammatical calculus, Ji obtained (α,β)-extensions of the formulas of Petersen and Stembridge.

We begin with a combinatorial setting of the (α,β)-Eulerian polynomials in terms of increasing binary trees. Based on an equivalent definition of Ji relying on the number of left-to-right minima and the number of right-to-left minima of a permutation, we observe that two particular leaves of an increasing binary tree, called the a-leaf and the b-leaf, play a special role. Then we move on to define the α-vertices and the β-vertices, and add the α-labels and the β-labels to certain internal vertices, while adopting the (x,y)-labeling for the leaves, as given in [3] for the bivariate Eulerian polynomials.

In fact, the two special leaves (the a-leaf and the b-leaf) can be considered as two poles to stretch a binary tree aligned on a horizontal line, which is reminiscent of the decomposition of a doubly rooted tree into a linear order of rooted trees in Joyal's proof of Cayley's formula [8]. More precisely, with these two special vertices at disposal, an increasing binary tree can be decomposed into a forest of planted increasing binary trees. Such a decomposition gives rise to a combinatorial interpretation of the γ-coefficients of the α-Eulerian polynomials in terms of forests of planted 0-1-2-plane trees. An interpretation in the permutation setting has been given by Ji-Lin [7] by devising a group action.

The idea of the labeling scheme for the (α,β)-Eulerian polynomials can be adapted to a grammar of Dumont related to the derangement polynomials. In this setting, we are led to a combinatorial interpretation of the γ-coefficients of the derangement polynomials and the q-derangement polynomials (with respect to the number of cycles), in terms of forests of planted increasing 0-1-2-plane trees, where the exponents of q are connected with the number of components of a forest. This topic has been extensively studied, see, for example, [9–15].

The grammatical labelings of increasing binary trees make it possible to give combinatorial interpretations of the sums involving the identities of Ji. We first realize that the number of interior peaks of a permutation can be read off from a labeling of increasing binary trees. For the rest, the decomposition of an increasing binary tree is the key ingredient all along.

2 The (α,β)-Eulerian polynomials

For $n \geq 1$, let $[n] = \{1,2,\ldots,n\}$. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$ of $[n]$, an index i $(2 \leq i \leq n)$ is called an ascent if $\sigma_{i-1} < \sigma_i$, and an index i $(1 \leq i \leq n-1)$ is called a descent if $\sigma_i > \sigma_{i+1}$. Let $\mathrm{asc}(\sigma)$ and $\mathrm{des}(\sigma)$ denote the number of ascents and the number of descents of σ, respectively.

Carlitz and Scoville [2] introduced an extension of the bivariate Eulerian polynomials, denoted by $A_n(x,y\,|\,\alpha,\beta)$, which are called the (α,β)-Eulerian polynomials by Ji [6]. They are defined by

$$A_n(x,y\,|\,\alpha,\beta) = \sum_{\sigma \in S_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} \alpha^{\mathrm{lrmax}(\sigma)-1} \beta^{\mathrm{rlmax}(\sigma)-1}, \tag{2.1}$$

where S_{n+1} is the set of permutations of $[n+1]$, $\mathrm{lrmax}(\sigma)$ and $\mathrm{rlmax}(\sigma)$ denote the number of left-to-right maxima and the number of right-to-left maxima of σ, respectively.

By taking complement of a permutation and exchanging the roles of x and y, Ji [6] presented an equivalent definition

$$A_n(x,y\,|\,\alpha,\beta) = \sum_{\sigma \in S_{n+1}} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)-1} \beta^{\mathrm{rlmin}(\sigma)-1}, \tag{2.2}$$

where $\mathrm{lrmin}(\sigma)$ and $\mathrm{rlmin}(\sigma)$ denote the number of left-to-right minima and the number of right-to-left minima of σ, respectively. The initial values of $A_n(x,y\,|\,\alpha,\beta)$ are given below,

$$\begin{aligned}
A_0(x,y\,|\,\alpha,\beta) &= 1, \\
A_1(x,y\,|\,\alpha,\beta) &= x\beta + y\alpha, \\
A_2(x,y\,|\,\alpha,\beta) &= xy\alpha + xy\beta + 2xy\alpha\beta + x^2\beta^2 + y^2\alpha^2.
\end{aligned}$$

Ji [6] found a context-free grammar for the (α,β)-Eulerian polynomials, which can be paraphrased as

$$G = \{a \to \alpha a y,\ b \to \beta b x,\ x \to xy,\ y \to xy\}. \tag{2.3}$$

By providing a grammatical labeling for permutations, it has been shown that the (α,β)-Eulerian polynomials can be generated by the above grammar.

Theorem 2.1 (Ji). *Let D denote the formal derivative of the grammar G. For $n \geq 0$, we have*

$$D^n(ab) = abA_n(x,y\,|\,\alpha,\beta). \tag{2.4}$$

As is well-known, permutations are in one-to-one correspondence with increasing binary trees, we find that endowed with a suitable labeling scheme increasing binary trees are conducive to a combinatorial understanding of the (α,β)-Eulerian polynomials. For this purpose, we shall introduce the (a,b,α,β)-labeling as described below.

2.1 The (a,b,α,β)-labeling

Let $n \geq 1$, and let T be an increasing binary tree on $[n]$, where $n \geq 1$. Consider the left child of the root. If it is a leaf, we call it the leftmost leaf of T. If not, we restrict to the left subtree of T and continue to seek the leftmost leaf. Eventually, we end up with the leftmost leaf of T. The rightmost leaf is defined in the same way. Now we label leftmost leaf of T by a and label the rightmost leaf of T by b.

Next, the α-vertices and the β-vertices are defined as follows. Each vertex on the path from the root to the a-leaf (other than the root and the a-leaf) is labeled by α, which we call an α-vertex. Each vertex on the path from the root to the b-leaf (other than the root and the b-leaf) is labeled by β, which we call a β-vertex. The rest of the leaves are labeled like the usual (x,y)-labeling, that is, a left leaf is labeled by x and a right leaf is labeled by y. It can be readily seen that a pair of sibling leaves labeled by x and y correspond to an interior peak of a permutation. For example, Figure 1 demonstrates an (a,b,α,β)-labeling of an increasing binary tree on [9], where the corresponding permutation reads

$$8\,4\,9\,6\,1\,2\,5\,3\,7.$$

The following theorem shows that the (α,β)-Eulerian polynomials have a combinatorial interpretation in terms of increasing binary trees. For an increasing binary tree T, we use $w(T)$ denote the weight of T with respect to the (a,b,α,β)-labeling, that is, the product of the grammatical labels. For instance, the weight of the increasing binary in Figure 1 equals $abx^4y^4\alpha^2\beta^3$.

In view of the correspondence between permutations and increasing binary trees, we

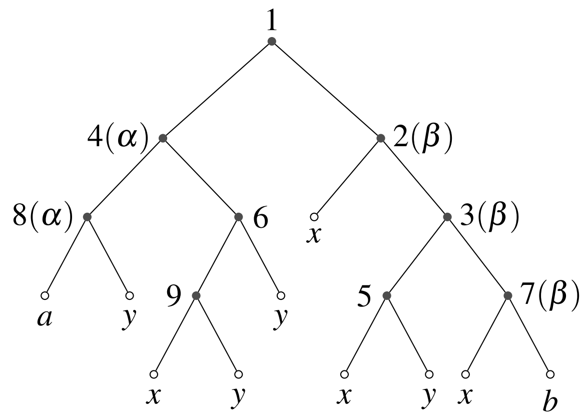

Figure 1: An example for the (a,b,α,β)-labeling.

see that the α-vertices together with the root are the left-to-right minima of the corresponding permutation, whereas the β-vertices together with the root are the right-to-left minima of the corresponding permutation. Then we arrive at the following combinatorial expansion. To be consistent with the meanings of x and y in the (a,b,α,β)-labeling, we use $A_n^*(x,y\,|\,\alpha,\beta)$ to denote $A_n(y,x\,|\,\alpha,\beta)$.

Theorem 2.2. *For $n \geq 1$, we have*

$$abA_n^*(x,y\,|\,\alpha,\beta) = \sum_T w(T), \tag{2.5}$$

where the sum ranges over the set of increasing binary trees on $[n+1]$ with the (a,b,α,β)-labeling.

2.2 A decomposition

The (a,b,α,β)-labeling leads us to consider a decomposition of an increasing binary tree into a forest of planted increasing binary trees, which can be used to divide the set of increasing binary trees into classes relative to the labeling scheme. By a planted increasing binary tree we mean an increasing tree structure consisting of a single root or a root with an increasing binary tree as a subtree. In the usual sense, a planted plane tree is either a single root or a plane tree for which the root has only one child.

Next, we introduce a decomposition of an increasing binary tree T on $[n] = \{1,2,\ldots,n\}$ into a forest of planted increasing binary trees rooted at the α-vertices and the β-vertices.

The resulting forest is called the supporting forest of T. For an increasing binary tree T on $[n]$, its supporting forest is on the set $[2,n] = \{2,3,\ldots,n\}$.

If we arrange the components of a supporting forest in the increasing order of their roots, then Figure 2 is an exhibition of the supporting forest of the increasing tree in Figure 1.

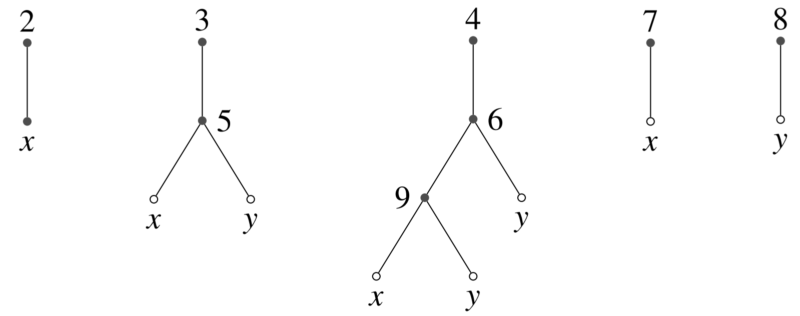

Figure 2: A supporting forest on $[2,n]$ with inherited labels.

The structure of a supporting forest can be used to divide the set of increasing binary trees on $[2,n]$ into classes whose total weight can be readily characterized. To this end, we define the weight of a supporting forest by the following rules. First, we suppress the leaf of a single root.

1. A single root is assigned the weight $x\beta + y\alpha$.
2. A root with a child has weight $\alpha + \beta$.
3. Any leaf has the weight (or label) inherited from the original increasing binary tree.

The updated labeling of a supporting forest is illustrated in Figure 3.

Since the root of a component of a supporting forest can be either an α-vertex or a β-vertex, we are led to the following expansion, where the underlying set of the supporting forests has been rescaled down to $[n]$.

Theorem 2.3. *For $n \geq 0$, $A_n^*(x,y\,|\,\alpha,\beta)$ equals the total weight of supporting forests on $[n]$.*

Now we further classify supporting forests via a group action. We say that two supporting forests are in the same class if one can be obtained from another by swapping a leaf

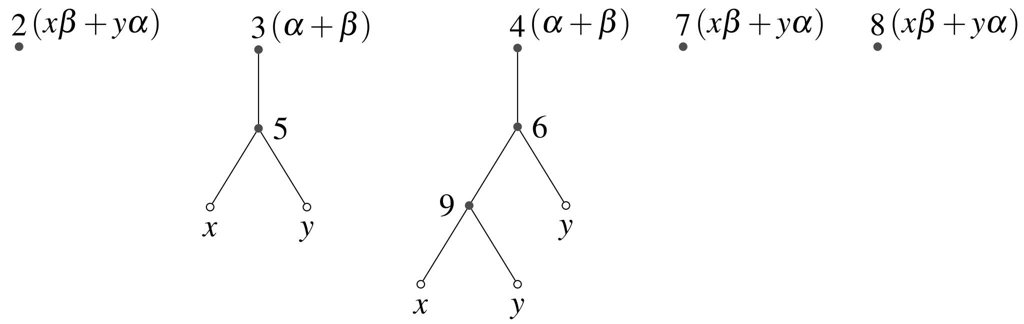

Figure 3: A supporting forest with updated labels.

with its non-leaf sibling. Therefore, such a class of supporting forests can be represented by a forest of planted 0-1-2 plane trees (without external leaves), bearing the following labeling rules:

1. A single root is endowed with a weight $x\beta+y\alpha$.
2. A root with a child has weight $\alpha+\beta$.
3. A degree one non-root vertex (a nonroot vertex with exactly one child) has weight $x+y$.
4. A leaf has weight xy.

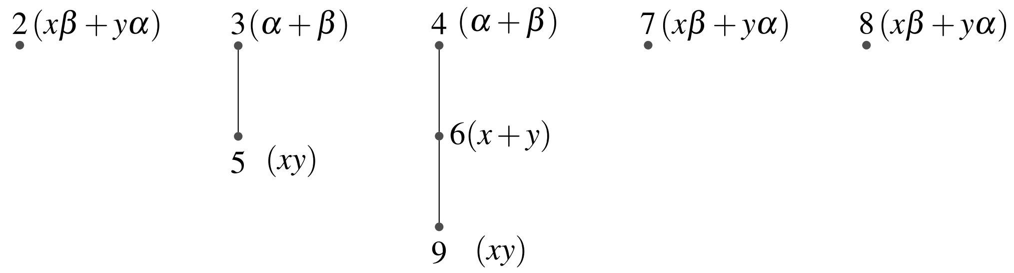

Figure 4: A forest of planted 0-1-2-plane trees.

Figure 4 is an illustration of a forest of planted 0-1-2-plane trees. The above classification implies the following expansion of the (α,β)-Eulerian polynomials.

Theorem 2.4. *For $n\geq 0$, $A_n^*(x,y\,|\,\alpha,\beta)$ equals the total weight of forests of planted 0-1-2-plane trees on $[n]$.*

2.3 The α-Eulerian polynomials

When $\alpha=\beta$, the (α,β)-Eulerian polynomials are called the α-Eulerian polynomials in [6], denoted by $A_n(x,y\,|\,\alpha)$. Likewise, we use $A_n^*(x,y\,|\,\alpha)$ to denote $A_n(y,x\,|\,\alpha)$. The labeling scheme of the corresponding trees is called the (a,b,α)-labeling. That is, all β-vertices are labeled by α as well. By a transformation of grammars, it is easy to see that these polynomials are γ-positive. Recall that by a γ-expansion, we mean an expansion in $x+y$ and xy. A bivariate polynomial is called γ-positive if the coefficients of the γ-expansion are all nonnegative. Evidently, the usual notion of γ-positivity for polynomials in x is equivalent to the bivariate formulation, whereas we do need both variables x and y as far as the grammar is concerned.

Ji and Lin [7] provided a combinatorial proof of the γ-coefficients by via a group action on permutations. With the help of the (a,b,α)-labeling, we obtain an alternative combinatorial interpretation of the γ-coefficients in terms of forests of planted 0-1-2-plane trees.

Setting $\alpha=\beta$, the previous weight assignment reduces to the following rules for the α-Eulerian polynomials. For a forest F of planted increasing 0-1-2-plane trees, we have the following rules:

1. A single root has weight $\alpha(x+y)$.
2. Other roots have weight 2α.
3. If a non-root vertex has only one child, it has weight $x+y$.
4. A leaf has weight xy.

Theorem 2.5. *For $n\geq 1$, the α-Eulerian polynomial $A_n^*(x,y\,|\,\alpha)$ has the γ-expansion*

$$\sum_F w(F),$$

where the sum ranges over forests of planted 0-1-2-plane trees on $[n]$.

2.4 The derangement polynomials

As a special case of the γ-expansion of the α-Eulerian polynomials, we come to the γ-expansion of the derangement polynomials.

Given a permutation σ in the cycle notation, assume that the minimum element of each cycle appears at the end, and the cycles are arranged in the increasing order of their minimum elements. For example, $(8\,4\,9\,6\,1)\,(2)\,(5\,3)\,(7)$ is a permutation of [9] in the cycle notation.

For an index $1 \leq i \leq n$, we call it an excedance of σ if $\sigma(i) > i$, or a drop if $\sigma(i) < i$, or a fixed point if $\sigma(i) = i$. Denote by $\mathrm{exc}(\sigma)$, $\mathrm{drop}(\sigma)$ and $\mathrm{cyc}(\sigma)$ the number of excedances, the number of drops and the number of cycles of σ, respectively. Let D_n be the set of permutations without fixed points. Then the derangement polynomials are defined by

$$d_n(x) = \sum_{\sigma \in D_n} x^{\mathrm{exc}(\sigma)},$$

see [1].

We can rely on the structure of forests of planted 0-1-2-plane trees to give a combinatorial interpretations of the γ-coefficients of the derangement polynomials, and the q-analogue with respect to the number of cycles, that is

$$d_n(x,y,q) = \sum_{\sigma \in D_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{drop}(\sigma)} q^{\mathrm{cyc}(\sigma)}.$$

Notice that a permutation without fixed points corresponds to a complete increasing binary tree without β-vertices whose left child is a leaf. A planted increasing binary tree is said to be fully planted if the root has a child that is not a leaf. By relabeling the root 1 by β and setting $\alpha = 1$ in the (a,b,α,β)-labeling, an increasing binary tree corresponding to a derangement can be decomposed into a forest of fully planted increasing binary trees for which the root of each tree is labeled by β. Then we take group action on a fully planted increasing 0-1-2-plane tree as follows. We label the root of each component by q. If a non-root vertex has degree one, then label it by $x+y$. A leaf is labeled by xy. Then the weight of a forest F of fully planted increasing 0-1-2-plane trees is defined to be the product of all the grammatical labels of F, denoted by $w(F)$. Then we get the following γ-expansion.

Theorem 2.6. *For $n \geq 1$, we have*

$$d_n(x,y,q) = \sum_{F} w(F), \tag{2.6}$$

where F ranges over forests of fully planted increasing 0-1-2-plane trees on $[n]$.

3 A labeling scheme for interior peaks

In this section, we give two labeling schemes of increasing binary trees in connection with interior peaks of a permutation, and we find combinatorial proofs of two identities of Ji.

Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, an index i $(2 \le i \le n-1)$ is called an interior peak if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$, and we follow the notation $M(\sigma)$ in [4] for the number of interior peaks of σ.

It turns out that the number of interior peaks can be read off from the (a,b,α,β)-labeling of the corresponding increasing binary tree. More precisely, a pair of sibling leaves labeled by x and y correspond to an interior peak of the permutation. An x-leaf whose sibling is not a y-leaf corresponds to an ascent of the permutation. Likewise, a y-leaf whose sibling is not an x-leaf corresponds to a descent of the permutation. Observe that the labels a and b play the role of preventing the first position and the last position from being counted as interior peaks.

First, let us consider the (α,β)-extension of Stembridge's identity [6, Theorem 1.8].

Theorem 3.1 (Ji). *For $n \ge 1$,*

$$\sum_{\sigma\in S_n} (xy)^{M(\sigma)} \left(\frac{x+y}{2}\right)^{n-2M(\sigma)-1} \alpha^{\mathrm{lrmin}(\sigma)-1}\beta^{\mathrm{rlmin}(\sigma)-1}$$
$$= \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} y^{n-\mathrm{des}(\sigma)-1} \left(\frac{\alpha+\beta}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2}. \qquad (3.1)$$

The case for $n = 1$ is trivial, so we assume that $n \ge 2$. To provide a combinatorial interpretation of the above relation, we shall give expansions of both sides in terms of forests of 0-1-2-planted plane trees, and will show that these two expansions are equinumerous, that is, they amount to the same total weights.

To reformulate the above relation in terms of trees, let $\mathscr{B}_n$ denote the set of increasing binary trees on $[n]$. Given $T \in \mathscr{B}_n$ endowed with the (a,b,α,β)-labeling, let $M(T)$ denote the number of vertices of T having two leaf children. As used in [3], $\mathrm{xleaf}(T)$ and $\mathrm{yleaf}(T)$ are referred to for the number of x-leaves and the number of y-leaves of T. Meanwhile, we write $N_\alpha(T)$ and $N_\beta(T)$ for the number of α-vertices and the number of β-vertices of T, respectively.

The relation (3.1) can be split into two parts. As for the left side, we have the following relation, where the set of forests of planted 0-1-2-plane trees on $[2,n]$ is denoted by $\mathscr{P}_n$.

Theorem 3.2. *For $n \geq 1$, we have*

$$\sum_{T\in\mathscr{B}_n} (xy)^{M(T)} \left(\frac{x+y}{2}\right)^{n-2M(T)-1} \alpha^{N_\alpha(T)}\beta^{N_\beta(T)} = \sum_{P\in\mathscr{P}_n} w(P), \tag{3.2}$$

where the sum ranges over the set of forests of planted 0-1-2-plane trees on $[2,n]$ with the following labeling rules and $w(P)$ stands for the weight of P:

1. *A single root is labeled by $(x+y)\frac{\alpha+\beta}{2}$.*
2. *The root of a component with at least two vertices is labeled by $\alpha+\beta$.*
3. *A degree one vertex other than the root is labeled by $x+y$.*

Proof. We begin with representing the sum on the left side over permutations in terms of a sum over increasing binary trees. Let T be an increasing binary tree of $\mathscr{B}_n$. We say that a leaf is proper if it is neither an a-leaf nor a b-leaf. In view of the (a,b,α,β)-labeling, $M(\sigma)$ corresponds to the number of internal vertices having two proper leaf children, whereas $n-2M(\sigma)-1$ equals the number of internal vertices having exactly one proper leaf child. Consequently, we are supposed to label T by the following rules, which we call the first modified (a,b,α,β)-labeling.

1. Label the leftmost leaf by a and label the rightmost leaf by b.
2. Any internal vertex on the path from the root to the a-leaf (other than the root) is labeled by α. Any internal vertex on the path from the root to the b-leaf (other than the root) is labeled by β.
3. For a pair of proper sibling leaves, we label the left leaf by x and the right leaf by y.
4. For a leaf whose sibling is not a proper leaf, we label it by $(x+y)/2$, no matter whether it is on the left or on the right.

For example, for the tree in Figure 1, the first modified (a,b,α,β)-labeling is demonstrated in Figure 5.

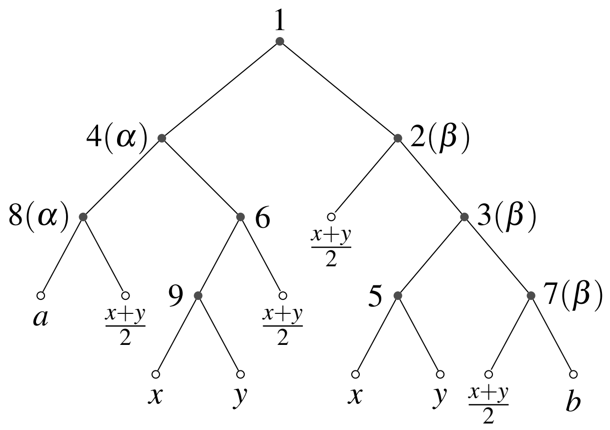

Figure 5: The first modified (a,b,α,β)-labeling.

We now process to compute the sum of weights over $\mathscr{B}_n$ by utilizing supporting forests. Let F be a supporting forest, that is, a forest of planted increasing binary trees on $[n]$. Let us characterize the set of trees T in $\mathscr{B}_n$ with supporting forest F on $[2,n]$. There are two choices for a planted increasing binary tree in F to belong to the left side (with the root being an α-vertex) or the right side (with the root being a β-vertex).

For a single root, it may originate from an α-vertex in T or a β-vertex in T. These two cases lead to the sum of weights

$$\frac{x+y}{2}\alpha+\frac{x+y}{2}\beta=(x+y)\frac{\alpha+\beta}{2}.$$

For a component of F containing at least two vertices, its root may originate from an α-vertex or a β-vertex, so the sum of weights equals $\alpha+\beta$.

Moreover, we can take a group action by swapping a proper leaf with its sibling that is not a leaf. Keep in mind that the a-leaf and the b-leaf no longer appear in F. Let $\mathrm{orb}(F)$ denote the orbit of F under this group action. Then let us compute the sum of weights of T with a supporting forest in $\mathrm{orb}(F)$. This quantity can be derived from a labeling of a representative of $\mathrm{orb}(F)$, that is a forest P of planted 0-1-2-plane trees.

Note that a proper left leaf with weight $(x+y)/2$ is paired with a proper right leaf with weight $(x+y)/2$, summing to a weight $x+y$. The above considerations suggest that we should comply with the rules as stated in the theorem. This completes the proof. ∎

Let us now turn to the sum on the right side of (3.1). A modification of the (a,b,α,β)-

labeling is needed, which we call the second modified (a,b,α,β)-labeling. In this case, both the α-vertices and the β-vertices are labeled by $(\alpha+\beta)/2$. For example, Figure 6 gives the modified labeling for the two trees in $\mathscr{B}_2$.

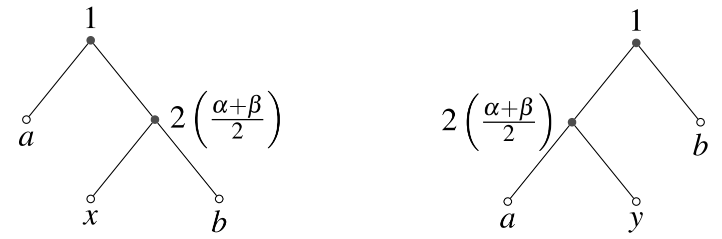

Figure 6: The second modified (a,b,α,β)-labeling.

At this point, the sum on the left of (3.1) can be recast in terms of the first modified (a,b,α,β)-labeling for increasing binary trees in $\mathscr{B}_n$, and we are left with the task to establish the following relation.

Theorem 3.3. *For $n \geq 1$,*

$$\sum_{T\in\mathscr{B}_n} x^{\mathrm{xleaf}(T)} y^{\mathrm{yleaf}(T)} \left(\frac{\alpha+\beta}{2}\right)^{N_\alpha(T)+N_\beta(T)} = \sum_{P\in\mathscr{P}_n} w(P), \tag{3.3}$$

where the sum ranges over $\mathscr{P}_n$ as in Theorem 3.2 and ditto the weight.

Proof. As before, we first consider the supporting forest of a tree in $\mathscr{B}_n$, and consider which trees in $\mathscr{B}_n$ share the same supporting forest F. Let T be an increasing binary tree in $\mathscr{B}_n$ with the supporting forest F.

For a single root in F, it may originate from an α-vertex with a right leaf child labeled by y, or a β-vertex with a left leaf child labeled by x. Given that all the α-vertices and β-vertices are labeled by $(\alpha+\beta)/2$, the two cases contribute a total weight of $(x+y)\frac{\alpha+\beta}{2}$, in accordance with the labeling of F.

For a component of F containing at least two vertices, its root may originate from an α-vertex or a β-vertex. Thus we get a total weight of

$$\frac{\alpha+\beta}{2}+\frac{\alpha+\beta}{2}=\alpha+\beta,$$

which coincides with the label of the root of F.

For other leaves of T, we consider the group action that swaps a proper leaf with its sibling that is an internal vertex. Strictly speaking, a proper x-leaf is paired with a proper y-leaf, giving a total weight of $x+y$. This group action gives rise to an orbit of F, which can be represented by a forest of planted 0-1-2-plane trees with weights as designated in the theorem. This completes the proof. ∎

Next, let us recall the (a,x,y,z)-labeling of an increasing binary tree with the x-leaves, y-leaves and z-leaves marking excedances, drops and fixed points of permutations respectively, see [5]:

1. If a β-vertex has a left leaf child, then this child is labeled by z, signifying a fixed point.
2. The rest of the leaves are labeled in the same manner as the (a,b,α,β)-labeling with a replaced by x and b replaced by a.

For example, with regard to the (a,x,y,z)-labeling, the increasing tree in Figure 1 corresponds to the following permutation in the cycle notation with the (a,x,y,z)-labels attached:

$$(8\,y\,4\,x\,9\,y\,6\,y\,1\,x)\ (2\,z)\ (5\,y\,3\,x)\ (7\,z)\ a.$$

Clearly, a derangement corresponds to an increasing binary tree without z-leaves.

We finish with a combinatorial proof of the following identity due to Ji, where D_n stands for the set of derangements of $[n]$.

Theorem 3.4 (Ji). *For $n \geq 1$,*

$$\sum_{\sigma\in S_{n+1}} (-1)^{\mathrm{des}(\sigma)} \left(\frac{1}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2} = \sum_{\sigma\in D_n} (-1)^{\mathrm{exc}(\sigma)}. \tag{3.4}$$

Proof. Let T be a tree in $\mathscr{B}_{n+1}$ on $\{0,1,\ldots,n\}$ with the (a,b,α,β)-labeling. Let F be the supporting forest of T. Consider the set of trees that share the same supporting forest as T. First, we observe a cancellation property. Note that F is a forest of planted increasing binary trees on $[n]$. On the other hand, we may regard T as an increasing binary tree endowed with the (x,y)-labeling for the Eulerian polynomials, that is, a left leaf is labeled by x and a right leaf is labeled by y. In the end, we set $x=-1$ and $y=1$.

We claim that a cancellation occurs when F contains a single root. If F contains a single root, then T has either an α-vertex with a y-leaf or a β-vertex with an x-leaf. These two possibilities create a pair of trees with the same supporting forest and opposite signs, here the sign of T is determined by the parity of the number of x-leaves. Moreover, such a pair of trees possess the same quantity

$$\mathrm{lrmin}(T) + \mathrm{rlmin}(T) - 2,$$

and hence we are led to a cancellation in the sum on the left of (3.4), which implies that the sum can be reduced to T whose supporting forests are fully planted.

Note that the (x,y)-labels of T are carried over to the forest F. This means that if two trees have the same supporting forest (without single roots), then they must have the same sign. Now we wish to compute the left side of (3.4) by classifying the supporting forests. Clearly, a supporting forest of k components generates 2^k trees in $\mathscr{B}_{n+1}$.

On the other hand, a supporting forest F can be viewed as an increasing binary tree on $[n]$ by gluing the component together. Up to now, it remains to make use of the fact that the labels carried over are precisely the same as the labels for the derangement polynomials with respect to the (a,x,y,z)-labeling, except for the rightmost y-leaf. Thus we may associate an x-label with an excedance and a y-label with a drop of the corresponding permutation. Finally, a special attention has to be paid to the rightmost y-leaf of T in $\mathscr{B}_{n+1}$ subject to the (a,x,y,z)-labeling. Since y is set to 1 at last, there are no worries. This completes the proof. ∎

To conclude, we remark that the above combinatorial argument yields a refinement of (3.4) by restricting the sum to

$$\mathrm{lrmin}(\sigma) + \mathrm{rlmin}(\sigma) - 2 = k.$$

Then the sum of the right side ranges over derangements with k cycles.

Conflict of Interest Statement: On behalf of all authors, the corresponding author states that there is no conflict of interest.

Acknowledgment. We wish to thank the referee for helpful comments. This work was supported by the National Science Foundation of China.

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